The finite automaton of an elementary cyclic set

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Rapports de Recherche

N° 824

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AVRIL 1988

RR 824

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March 23, 1988

Abstract

In recent work, we introduced the notion of elementary cyclic set of equations on first-order terms. These are non-unifiable sets of equations, but that are non-unifiable in a minimal sense. We proved that a minimum equational deduction was canonically associated to these equations. Here we introduce and establish some properties of a finite automaton associated to these sets of equations. This automaton describes the structure of the search-tree of some higher-order unification problems, whose functional equations define an elementary cyclic set by stripping the higher-order structure. The idea underlying these studies is that non-unifiable first-order equations can be solved at higher-order. To this end, the canonical deduction and the finite automaton are essential tools. The motivation of this work lies in the problem of higher-order type inference for programs.

L'Automate Fini d'un Ensemble Cyclique Elémentaire

Résumé

Dans un travail récent, nous avons introduit la notion d'ensemble cyclique élémentaire d'équations entre termes du premier ordre. Un tel ensemble est non-unifiable, mais de façon minimale. Une déduction équationnelle minimum est canoniquement associée à un tel ensemble. Dans ce rapport, nous introduisons et établissons quelques propriétés d'un automate fini associé à ces ensembles d'équations. Cet automate décrit la structure de l'arbre de recherche de certains problèmes d'unification à l'ordre supérieur, dont les équations fonctionnelles définissent un ensemble cyclique élémentaire en supprimant la structure à l'ordre supérieur. L'idée principale sous-jacente est que ces ensembles non unifiables au premier ordre peuvent l'être à l'ordre supérieur. A cette fin, la déduction canonique et l'automate fini sont des outils indispensables. La motivation de ce travail réside dans le problème de l'inférence des types à l'ordre supérieur.

1 Introduction

Solving functional equations is the basic process of theorem-proving procedures in higher-order logic. The underlying calculus is the simply typed λ-calculus that we assume known [1]. Up to date the most efficient search procedure for such equations is higher-order unification as precised by Huet [9], reference that we assume known. Naturally, the problem is undecidable [6] and, up to date, little is known about the decidable subclasses. We may cite the results of Zaionc [14], for some regular subclasses. Our approach to higher-order unification is motivated by the higher-order type inference problem. The results are valid for terms over an arbitrary signature. For convenience, we assume throughout the paper that the signature contains a single binary infix function symbol, noted by an arrow (thus revealing the origin of the problem). Functional equations involved in the type inference problem have the following important property. The set $\mathcal{S}$ of *principal* head variables of a set of equations E is defined by: if M is a member of some equation in E, then the head variable of M is either a free variable Φ or a constant (here, the arrow). In the former case, $\Phi \in \mathcal{S}$, in the latter we recursively apply these two alternatives to the arguments of the constant. Consider now the λ-terms arguments of variables from $\mathcal{S}$, then their set of free variables is disjoint from $\mathcal{S}$, and, equally important, these free variables have a unique occurrence. Also, the first-order approximation of such equations, obtained by stripping the λ-structure, connects first-order to higher-order equations. A detailed analysis of the first-order equations will be valuable in obtaining properties of the solvability of higher-order equations.

This paper is a companion paper of [10], and we assume known its notions and results. In essence, there we proved that the set of occur-checks preventing unifiability of a set of equations splits into *primitive* and *derived* occur-checks. The notion of primitive occur-check is sequential: there exists a canonical equational deduction of a cyclic equation that linearly orders the variable occurrences involved in a primitive occur-check. This equational deduction plays a fundamental rôle in the present paper.

A semi-decision procedure for higher-order unification iteratively performs both *imitations* and *projections* on equations. The structure of the set of imitations motivates this study. If the set of higher-order equations has an associated set of first-order equations which is elementary cyclic, then this structure is described by a finite automaton. The direct road towards this automaton introduces it via the minimum deduction of [10]. Under some imitation, this deduction is transformed into a new one under a reduction process. Sequentiality of deductions is preserved by this reduction. Intuitively, successive imitations stepwise erase the *auxiliary* deductions [10]. When this is done, the set of deductions reachable by imitation is finite. They represent *states* of the automaton, with *transitions* the imitations. For the cognoscenti, strict equations play an important rôle in the present work, the alert reader who knows higher-order unification will recognize that they correspond to flexible-flexible pairs.

The first section introduces the reduction of deductions under the imitations. The second one defines the automaton, proves its properties and provides some examples. We illustrate the above informal explanations by a concrete example. The single equation

$$E : \Phi(X) = \Phi(Y) \rightarrow \Psi$$

defines by stripping the equation $\phi = \phi \rightarrow \psi$, which is non-unifiable, due to a unique positive occur-check. The equation E has the properties required above: the variables of the arguments (X and Y) are disjoint from the principal variables (Φ and Ψ). Further both X and Y have a single occurrence.

It follows that by projection the equation E is solved: if Φ is the identity function, E becomes $X = Y \to \Psi$, trivially solvable. Now under successive imitations: $\sigma_1(\Phi) = \lambda\alpha.\Phi_1(\alpha) \to \Phi_2(\alpha)$, $\sigma_2(\Phi_1) = \lambda\alpha.\Phi_3(\alpha) \to \Phi_4(\alpha), \ldots$, the equation E gives

$$\begin{cases} \Phi_{2n+1}(X) &= \Phi_{2n+1}(Y) \to \Phi_{2n+2}(Y) \\ \Phi_{2n+2}(X) &= \Phi_{2n}(Y) \\ \Phi_{2n}(X) &= \Phi_{2n-2}(Y) \\ & \vdots \\ \Phi_2(X) &= \Psi \end{cases}$$

Forgetting the flexible-flexible equations, this system reproduces itself and its finite automaton possesses a single state s and a single transition from s to s. The set of solutions is the regular language

$$\lambda\alpha.\alpha, \qquad \lambda\alpha.\alpha \to \Psi(\alpha), \qquad \lambda\alpha.(\alpha \to \Psi(\alpha)) \to \Psi(\alpha) \ldots$$

where Ψ denotes an arbitrary function.

2 Development of Minimum Deductions

The λ-terms are assumed to be in $\beta\eta$-long normal form. If each bound variable occurs as argument of some free variable, then to a set of functional equations is associated a set of first-order equations as follows:

- $\mathrm{Strip}(\lambda\overline{\alpha}.\Phi M_1 \cdots M_k) = \Phi$, Φ a free variable ($\lambda\overline{\alpha}$ denotes a sequence of abstractions);
- $\mathrm{Strip}(\lambda\overline{\alpha}.F M_1 \cdots M_l) = F(\mathrm{Strip} M_1) \cdots (\mathrm{Strip} M_l)$, F a constant.

Higher-order unification grows a search tree by imitations and projections (cf. [8]). Given a search tree for this set of higher-order equations, its imitation structure for its principal head variables is reflected by a tree on the associated set of first-order equations. This motivates the following definition:

Definition 2.1 *Let E be a set of equations.*

- *A substitution σ is elementary iff $dom(\sigma)$ is a singleton $\{\phi\}$ and $\sigma(\phi) = \psi \to \theta$ where the variables ϕ, ψ and θ are all distinct.*
- *The elementary substitution σ is acceptable for E iff ϕ is the left-hand side of some non-strict equation in E and ψ, θ do not belong to* $\mathrm{Vars}(E)$.
- *An acceptable elementary substitution for E is called an imitation of E.*
- *If E is an elementary cyclic set, an imitation is called a development of E.*

Let S be some elementary cyclic set and $\mathcal{D}$ be its minimum deduction. Under a development σ of S, this deduction $\mathcal{D}$ can be transformed into a new deduction $\mathcal{D}'$. This deduction development introduces the derived elementary cyclic set of S under this development. We will prove that the

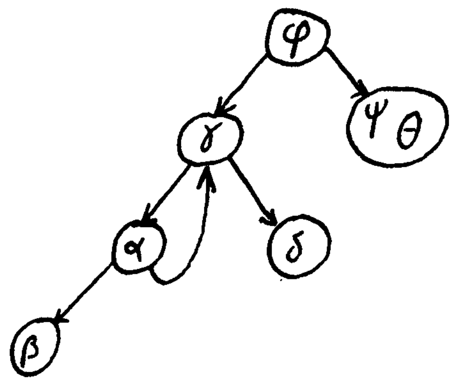

Figure 1: The unification graph $\mathcal{G}_E$.

set of axioms of the developped deduction *defines* the derived elementary cyclic set included in $\mathrm{Simpl}(\sigma(S))$. Let us see an example:

$$E \quad \begin{cases} \phi &= \gamma \to \psi \\ \phi &= (\alpha \to \delta) \to \theta \\ \alpha &= \beta \to \gamma \end{cases}$$

The set E is an elementary cyclic set whose minimum deduction $\mathcal{D}$ is:

$$(su)\,\frac{(d)\,\dfrac{\phi = \gamma \to \psi \quad \phi = (\alpha \to \delta) \to \theta}{\gamma = \alpha \to \delta} \qquad \alpha = \beta \to \gamma}{\alpha = \beta \to (\alpha \to \delta)}$$

Under the development $\sigma(\phi) = \phi_1 \to \phi_2$, we have

$$\mathrm{Simpl}(\sigma(E)) = \{\phi_1 = \gamma,\ \phi_2 = \psi,\ \phi_1 = \alpha \to \delta,\ \phi_2 = \theta,\ \alpha = \beta \to \gamma\}$$

The deduction becomes:

$$(su)\,\frac{(t)\,\dfrac{(s)\,\dfrac{\phi_1 = \gamma}{\gamma = \phi_1} \quad \phi_1 = \alpha \to \delta}{\gamma = \alpha \to \delta} \qquad \alpha = \beta \to \gamma}{\alpha = \beta \to (\alpha \to \delta)}$$

The derived elementary cyclic set contains the three equations $\phi_1 = \gamma$, $\phi_1 = \alpha \to \delta$, $\alpha = \beta \to \gamma$. Under the development $\sigma'(\alpha) = \alpha_1 \to \alpha_2$ we have

$$\mathrm{Simpl}(\sigma'(E)) = \{\phi = \gamma \to \psi,\ \phi = ((\alpha_1 \to \alpha_2) \to \delta) \to \theta,\ \alpha_1 = \beta,\ \alpha_2 = \gamma\}$$

We get the derived deduction:

$$(t)\,\frac{\alpha_2 = \gamma \qquad (d)\,\dfrac{\phi = \gamma \to \psi \quad \phi = ((\alpha_1 \to \alpha_2) \to \delta) \to \theta}{\gamma = (\alpha_1 \to \alpha_2) \to \delta}}{\alpha_2 = (\alpha_1 \to \alpha_2) \to \delta}$$

The derived elementary cyclic set does not contain $\alpha_1 = \beta$.

Proposition 2.1 *Let $\mathcal{D}$ be the minimum deduction of the elementary cyclic set S. Let σ be some development of S, $\mathcal{D}$ can be transformed into a cyclic deduction $\mathcal{D}'$ whose axioms are equations from* $\mathrm{Simpl}(\sigma(S))$.

Proof. Assume that the development σ substitutes $\phi_1 \to \phi_2$ for ϕ; ϕ_1, ϕ_2 fresh variables. The set S contains at least one equation $\phi = \tau$, $|\tau| > 0$.

If $\mathcal{E} \vdash \psi = C[\phi]$ is some subdeduction of $\mathcal{D}$, we can replace this occurrence of ϕ by $\phi_1 \to \phi_2$ throughout $\mathcal{E}$ and get a valid deduction. This is immediate if this occurrence belongs to some right-hand side. Otherwise this occurrence of ϕ necessarily is the left-hand side of some strict equation $\phi = \theta$ in $\mathcal{A}(\mathcal{D})$, premiss of an instance of the symmetry rule. We merely replace this instance of (s) by $\theta = \phi_1 \to \phi_2$. This observation will be implicitely used below, where we denote the new deduction by $\overline{\mathcal{E}}$.

If S contains a single cyclic equation $\phi = C[\phi]$, the existence of $\mathcal{D}'$ is immediate. Otherwise, we locally transform $\mathcal{D}$ according to the occurrences of ϕ. Firstly we reduce subdeductions involving a non-strict axiom from S whose left-hand side is ϕ, by cases on the rule that eliminates ϕ. Secondly, it will remain the cases of two right-hand side occurrences of ϕ, eliminated by the same rule. Notice that in each case we have a symmetry ϕ_1/ϕ_2. Further, the reduced deductions that we build can create a redex with their context in $\mathcal{D}$. We left the reader check that the deduction can be reduced according to the rules given in [10]. The reductions possibly modify the conclusion of the deduction, but remind that all we want is a cyclic deduction. Finally, some reductions involve a context, say $C[\theta]$. When $C[\theta] = \theta$, the rule whose one premiss contains $C[\theta]$ must be replaced by a (t)-rule, according to the reductions given in [10]. For the convenience of the reader, the list of reductions is given in Appendix 1. □

For a set of equations derived from the elementary cyclic set S by successive developments, we introduce a labeling of its equations. Labels are pairs (n, O), n an integer, O an occurrence. Equations in S are uniquely labeled by $(1,\epsilon), \ldots, (m,\epsilon)$ where $m = |S|$ and ϵ is the empty occurrence. We define the function vertex V from equations to vertices in $\mathcal{G}_S$ first by $V(k,\epsilon) = V(\phi)$ iff $(k,\epsilon) : \phi = \tau \in S$. Assume that the set S' is derived from S and that both labels and vertices are defined on equations in S'. Under a development σ, the set $\mathrm{Simpl}(\sigma(S'))$ necessarily contains new equations. Their labels are computed as follows. Let $(n, O) : \phi = \tau \in S'$ be an equation such that $\sigma(\phi) = \phi_0 \to \phi_1$ and $\sigma(\tau) = \tau_0 \to \tau_1$. Then $\mathrm{Simpl}(\sigma(S'))$ contains $(n, O.0) : \phi_0 = \tau_0$ and $(n, O.1) : \phi_1 = \tau_1$ with $V(n, O.0) = v_0$ and $V(n, O.1) = v_1$ where v_0 (resp. v_1) is the left (resp. right) successor of $V(n, O)$. The labels and vertices of other equations are unmodified.

Lemma 2.2 *Let $S' \subseteq \mathrm{Simpl}(\sigma(S))$ be the set of axioms of $\mathcal{D}'$, defined by the deduction $\mathcal{D}$ of the elementary cyclic set S under some development. Then, modulo a possible renaming of one of the new variables introduced by σ, S' is an elementary cyclic set, whose minimum deduction is $\mathcal{D}'$.*

Proof. The graph $\mathcal{G}_{S'}$ contains at least one cycle from the existence of the deduction $\mathcal{D}'$ by Proposition 2.1. The labels of equations establish that the abstract graph underlying $\mathcal{G}_{S'}$ is a subgraph of the abstract graph underlying $\mathcal{G}_S$. Hence $\mathcal{G}_{S'}$ contains a unique cycle.

Let $\sigma(\phi) = \phi_1 \to \phi_2$. If the axioms of $\mathcal{D}'$ include two equations $\psi_1 = C_1[\phi_1 \to \phi_2]$ and $\psi_2 = C_2[\phi_1 \to \phi_2]$ and if all minimal paths from $V(\phi)$ to the cycle start with the same successor of $V(\phi)$, say $V(\phi), V(\phi_1), \ldots$, then the occurrences of ϕ_2 in S' must be linearized in order to get an elementary cyclic set from S'. With this proviso, assume now that either some equation in S' is redundant or that some occurrence in S' can be linearized. Both cases imply that we have an equational deduction $\mathcal{D}''$, which is smaller than $\mathcal{D}'$: it does not use some equation in S' or it does not use an occurrence of some needed variable. We lift back this deduction to a cyclic deduction of S. This is possible by the labeling of axioms and the fact that if ϕ is imitated, when ϕ_1 or ϕ_2 occurs in some non-strict right-hand side, we have a context of the form $C[\phi_1 \to \phi_2]$. The lifting is done by a rewriting process, presented by reduction rules, similar to those used in the proof of Proposition 2.1. While giving them, we use the symmetry ϕ_1/ϕ_2 and the facts that right-hand side of the premisses of (d)-rules have a positive size, and right-hand side of the right premisse of a (su)-rule also has a positive size (cf. [10]). We present the rules by cases on the premisses, either we have two, one or no axioms. Subcases correspond to the various rules. Some contractum involve a (d)-rule with premiss of the form $\psi = C[\phi]$, when $C[\phi] \equiv \phi$, the contractum should be replaced by an instance of the (t)-rule. As in the proof of Proposition 2.1, the list of reductions is given in Appendix 2.

The existence of a deduction strictly smaller than $\mathcal{D}$ implied by these reduction applied to $\mathcal{D}''$ contradicts $\mathcal{D}$ minimum for S. This proves that $\mathcal{D}'$ defines an elementary cyclic set and that $\mathcal{D}'$ is its minimum deduction. □

We introduce a complexity measure on deductions. Let $e : \phi = \tau$ be an equation in S, and c be the cycle of $\mathcal{G}_S$, the notation $e \in c$ means $V(\phi)$ cyclic. We say that the equation e is cyclic. Remind that if ϕ is needed there exists at least one path from $V(\phi)$ to the cycle c.

- If $V(\phi) \notin c$, $d(e) = min\{d(V(\phi), v) \mid v \in c\}$ where $d(v, v')$ is the length of a minimal path between v and v'.
- If $V(\phi) \in c$, $p(e) = |O_C|$, with $e : \phi = C[\psi]$ (remind that in this case, the right-hand side of e contains a unique needed variable [10]).
- If $V(\phi) \in c$, $q(e) = d(V(\phi), v)$, where v is either the first shared vertex of the cycle c below $V(\phi)$, or $V(\phi)$ itself when either $V(\phi)$ is shared or there exist no shared vertices along c.

Notice that $q(e)$ is the distance along the cycle, i.e. the length of $p = V(\phi), \ldots, v$, p subpath of c (otherwise there would exist two cycles in the graph $\mathcal{G}_S$). Let $C(S)$ be the subset of the non-strict cyclic equations from S whose left-hand side does not occur by some edge not belonging to the cycle. The complexity $\mu(S)$ of the elementary cyclic set S is the quad:

$$\mu(S) = \left(\sum_{e \in S} d(e), \sum_{e \in c} p(e), \sum_{e \in C(S)} p(e), \sum_{e \in c} q(e) \right)$$

with $(t_1, t_2, t_3, t_4) < (t'_1, t'_2, t'_3, t'_4)$ iff

- either $t_1 < t'_1$,
- or $t_1 = t'_1 > 0$ and $t_2 < t'_2$,
- or $t_1 = t'_1 > 0$, $t_2 = t'_2$ and $t_3 < t'_3$,
- or $t_1 = t'_1 > 0$, $t_2 = t'_2$, $t_3 = t'_3$ and $t_4 < t'_4$.

This ordering is well-founded on quads whose first component is positive. This condition for $\mu(S)$ is equivalent to the existence of non (d)-free auxiliary deductions in $\mathcal{D}$, the minimum deduction of S, or equivalently, to the existence of initial vertices in $\mathcal{G}_S$. Let σ be some development of the elementary cyclic set S. By abuse of notation, we call the set $\mathcal{A}(\mathcal{D}') \cap \mathrm{Simpl}(\sigma(S))$ a development of S. If S' is developped from S under σ, we note $S \vdash^1_\sigma S'$, $S \vdash S'$ means that there exists a sequence of successive developments from S to S'. We write $S \vdash^n S'$ if S' is derived from S under a sequence of n successive developments. The independance of n with respect to a particular choice of substitutions is easily checked: n is the size of the substitution from S to S', the size of a substitution being the sum of the sizes of its range.

Lemma 2.3 *Assume that the graph of the elementary cyclic set S possesses initial vertices. If S' is a development of S, we have $\mu(S') < \mu(S)$.*

Proof. Let $\mu(S') = (t'_1, t'_2, t'_3, t'_4)$ and $\mu(S) = (t_1, t_2, t_3, t_4)$. Assume that the development σ of S substitutes $\phi_1 \to \phi_2$ for ϕ. If $V(\phi)$ is non-cyclic, we have $t'_1 < t_1$. Otherwise, we have $t'_1 = t_1 > 0$. This follows from the fact that the minimal distance from $V(\psi)$ to c, $V(\psi)$ non-cyclic, is unmodified by an imitation of a cyclic needed variable, as the abstract graphs are included one in the other. The inequality $t_1 > 0$ follows from the assumption that $\mathcal{G}_S$ possesses initial vertices. From [10], we know that ϕ possesses exactly two occurrences, among them one as left-hand side of a non-strict equation. We reason by cases on the other occurrence $e' : \psi = C[\phi]$:

1. It is member of a strict equation; i.e. we have $C[\phi] = \phi$ and there exists exactly two equations involving ϕ, say $\phi = \tau_1 \to \tau_2$, $\phi = \psi$ or $\psi = \phi$ in S. This gives, say $\phi_1 = \tau_1$, and $\psi = \phi_1 \to \phi_2$ in S', as a cyclic equation possesses only one needed occurrence in its right-hand side [10]. Therefore $t'_2 = t_2$.
   - If $V(\phi)$ is shared, then ψ necessarily occurs by some non-cyclic edge in $\mathcal{G}_S$, while ϕ does not, hence $t'_3 < t_3$.
   - If $V(\phi)$ is not shared, then $t'_3 = t_3$, while $t'_4 < t_4$, as there exists at least one shared vertex by $t_1 = t'_1 > 0$.
2. If $e' : \psi = C[\phi]$, $C[\phi] \neq \phi$, we still have two subcases:
   - $V(\psi)$ non cyclic implies $t'_2 < t_2$.
   - $V(\psi)$ is cyclic. We have $V(\psi) \neq V(\phi)$. By the structure of needed cyclic vertices [10], we have $V(\phi)$ unshared. Hence $t'_3 < t_3$.

This concludes the proof. □

We give some intuition underlying this complexity measure: under successive developments, one eventually imitates some non-cyclic equation. Hence all sequences of successive developments eventually gives (d)-free deductions, i.e. without auxiliary deduction. An elementary cyclic set whose minimum deduction is (d)-free will be called a *pure* elementary cyclic set. We have proved that all sequences of developments eventually creates pure elementary cyclic sets. As is easily seen, any development of a pure elementary cyclic set S gives a pure elementary cyclic set S'. Further the abstract graphs $\mathcal{G}_S$ and $\mathcal{G}_{S'}$ are equal. For example, the set E of section 1, under the development σ gives a pure elementary cyclic set, while the development σ' defines a non-pure one. But the only possible development of this set does give a pure elementary cyclic set.

3 The Automaton of an Elementary Cyclic Set

We define a finite automaton associated to an elementary cyclic set S. This automaton must not be confused with the automaton of the regular forest defined by S [3]. Its states are equivalence classes of derived elementary cyclic sets, the transitions being developments.

We first need some properties of pure elementary cyclic sets and their developments. Let S be an elementary cyclic set. For ϕ a needed variable of S, $\pi(\phi)$ denotes half the number of distinct marked occurrences of ϕ in the axioms of $\mathcal{D}$, the minimum deduction of S. Remind that occurrences of variables in $\mathcal{D}$ are marked and note that two distinct occurrences of the same equation in $\mathcal{D}$ defines two occurrences of its left-hand side (and of the right-hand side marked variable if this variable is the same for the two occurrences). The set of needed variables of S is noted $\mathcal{N}(S)$.

Lemma 3.1 *Let S be an elementary cyclic set and S' be a pure elementary cyclic set with $S \vdash S'$. We have $|S'| = |\mathcal{N}(S')| = \sum_{\phi\in\mathcal{N}(S)} \pi(\phi)$. Let n be the number of equations in S' and l be the length of its cycle. In a sequence of developments of length $N = n\times(l-1)+(n-1)\times l+1$, starting from S', any needed variable of S' is developped at least l times.*

Proof. We first establish the second equality by induction on the length m of sequences of developments from S to S'. The equality is true if $S' = S$ as $\pi(\phi) = 1$ for each needed variable [10]. Assume it true for m and let S'' be such that $S \vdash^1_\sigma S'' \vdash^m S'$, with $\sigma(\phi) = \phi_1 \to \phi_2$. It is easily seen from the reductions in Appendix 2 that $\sum_{\phi\in\mathcal{N}(S)} \pi(\phi) = \sum_{\phi\in\mathcal{N}(S'')} \pi(\phi)$. Hence, by induction hypothesis $|\mathcal{N}(S')| = \sum_{\phi\in\mathcal{N}(S)} \pi(\phi)$.

To prove the first equality, remind first that the size $|e|$ of an equation e is the size of its right-hand side. From [10], we have $\sum_{e\in S'} |e| = l$. Let $v_1,\ldots,v_l$ be the sequence of vertices of c, the cycle of S', and let $v_{i_1},\ldots,v_{i_k}$ be the sequence of needed vertices. The set $\mathcal{N}(S')$ can be ordered as follows:

$$(\underbrace{\alpha^1_1,\ldots\alpha^1_{j_1}}_{v_{i_1}},\underbrace{\alpha^2_1,\ldots\alpha^2_{j_2}}_{v_{i_2}},\ldots,\underbrace{\alpha^k_1,\ldots\alpha^k_{j_k}}_{v_{i_k}})$$

with the convention that:

- $\alpha^p_i = \alpha^p_{i+1} \in S'$, $V(\alpha^p_i) = v_{i_p}$, for $p = 1,\ldots,k$, $i = 1,\ldots,j_p-1$.
- $\alpha^p_{j_p} = C_p[\alpha^{p+1}_1] \in S'$, $C_p[\alpha^{p+1}_1] \neq \alpha^{p+1}_1$, for $p = 1,\ldots,k$, with $\alpha^{k+1}_1 =_{def} \alpha^1_1$.

In other words, each variable occurs exactly twice in S', and the number of needed variables equals the number of equations: $\sum_{i=1}^k((j_i-1)+1) = |S'| = |\mathcal{N}(S')|$.

For a pure elementary cyclic set and its developments, we may consider that the set of variables remains fixed: if ϕ is developped in $\phi_1 \to \phi_2$, then one and only one of ϕ_1, ϕ_2 belongs to the developped elementary cyclic set. With this convention, we now proceed to prove that a (cyclic) variable in turn visits each cyclic vertex in a development sequence of length N. The problem is equivalent to the following one: given a ring of n boxes containing l pebbles, a firing rule states that at each time clock, one and only one pebble can move from a box b to its neighbour b', clockwise. We say that box b has been fired. How many firing are needed so that each box fires l times at least? The answer comes from the worst configuration: only one box, say b, is fired exactly $l-1$ times, while all other boxes are fired at least l times, this is achieved by putting l pebbles in the neighbour b' of b. Firstly, one pebble fires $n \times (l-1)$ times (hence $l-1$ firing for b), secondly, all pebbles fires $n-1$ times. □

To any rule instance in the minimum deduction $\mathcal{D}$ corresponds a path from the vertex of the variable eliminated by this rules instance to the cycle c. We introduce the alphabet $\mathcal{A}$ of an elementary cyclic set S: it is the set of variables ϕ_q, $\phi \in \mathcal{N}(S)$, q a prefix of the path p, ranging from the void path to p, where ϕ is eliminated by some rule whose associated path is p, or ϕ is the main variable of $\mathcal{D}$ (and p is the trivial path). Following the observation about variables in pure elementary cyclic sets in the proof of Lemma 3.1, developments are assumed to belong to $\mathcal{A}$. Note that $|\mathcal{A}| = \sum_{\phi \in \mathcal{N}(S)} \pi(\phi)$. This alphabet will be the alphabet of the finite automaton associated to S.

We now define an equivalence relation on derived sets. Two derived elementary cyclic sets are equivalent when their sets of equation labels are equal modulo the cycle. Let (k, O) be some label and $v = V(k, O)$ be its vertex, the occurrence O_v is defined to be ϵ, the void occurrence, if v is a non-cyclic vertex, and the occurrence of the cycle c started at v if v is cyclic, i.e. $c = (v, O_v)$ in the notation of [10]. The set of labels of an elementary cyclic set S is denoted by $\mathcal{L}(S)$.

Definition 3.1 *Let S be an elementary cyclic set, and $S \vdash S'$, $S \vdash S''$. Then $S'' \leq S'$ iff there exists an integer m such that*

$$(k, O) \in \mathcal{L}(S') \quad \text{iff} \quad (k, O.(O_v)^m) \in \mathcal{L}(S'')$$

where $v = V(k, O)$. We note $S' \sim S''$ when $S' \leq S''$ or $S'' \leq S'$.

The notation $S' \sim_m S''$ explicits the value of the integer in the defintion of the equivalence of the two sets. We now establish some properties of the equivalence classes. Intuitively, two equivalent elementary cyclic sets are equal up to some number of unfoldings of the cycle under imitations. The last two propositions, stating that developments act on equivalence classes and that the number of these classes is finite, allow us to introduce the automaton of an elementary cyclic set.

Proposition 3.2 *Let S be an elementary cyclic set, and S', S'' be two sets derived from S.*

1. *If $S' \sim_m S''$ and the labels of these sets are not equal, both S' and S'' are pure elementary cyclic sets and $S' \vdash^{m \times (n \times l)} S''$ or $S'' \vdash^{m \times (n \times l)} S'$.*
2. *If $S' \sim S''$, there exists a bijection σ between $\mathcal{N}(S')$ and $\mathcal{N}(S'')$ such that $\alpha = C'[\beta_1, \ldots, \beta_k]$ is an equation of S' iff $\sigma(\alpha) = C''[\sigma(\beta_1), \ldots, \sigma(\beta_k)]$ is an equation of S'', where $C' \sim C''$ and the lists of needed occurrences in right-hand sides are $\beta_1, \ldots \beta_k$ and $\sigma(\beta_1), \ldots, \sigma(\beta_k)$ respectively.*
3. *If $S'_0 \sim S''_0$ and ϕ_q is a development of S'_0, then ϕ_q is also a development of S''_0. Let S'_1 and S''_1 be the elementary cyclic sets defined by ϕ_q applied to S'_0 and S''_0, then $S'_1 \sim S''_1$.*
4. *The number of equivalence classes of derived sets is finite.*

Proof. A first observation is that labels are in bijective correspondance with equations. First proposition: it follows from the definition of O_v that all vertices of labels, hence of equations, are cyclic. But this is the definition of a pure elementary cyclic set. For its second part, assume that $S' \leq S''$. It suffices to establish its truth for $m = 1$ by transitivity of the relation $\vdash$. This in turn follows from the decomposition of the needed cyclic vertices in Lemma 3.1. For each non-strict equation $\alpha = C[\beta]$ in this decomposition, C non-trivial, we perform $|O_C|$ sequences of n

developments as follows: α is developped, the previous equation becomes, say, $\gamma = C'[\alpha_1 \rightarrow \alpha_2]$, then γ is developped, etc. By Lemma 3.1 $\sum_{S'} |O_C| = l$. The elementary cyclic set S''' that we get has its labels equal to those of S''. By the first part of the proof of proposition 2 below, we have $S'' = S'''$.

For proposition 2, if the labels are equal, we have $S' \subseteq \mathrm{Simpl}(\sigma'(S))$ and $S'' \subseteq \mathrm{Simpl}(\sigma''(S))$, for two sequences of developments σ' and σ''. From this we have $|\sigma'| = |\sigma''|$. We use induction on $|\sigma'|$. If $|\sigma'| = 0$, then $S' = S'' = S$. Otherwise, let ϕ_q be the first development of σ'. If this is also the first development of σ'', then we conclude by induction hypothesis. Otherwise, ϕ_q appears necessarily in σ''. Being initially enabled, there exists a substitution – sequence of developments – σ, starting by ϕ_q and equal to σ'' up to the first occurrence of ϕ_q. We may conclude by induction hypothesis.

If labels are distinct, we use the same sequence of developments used to establish proposition 1 with $m = 1$. The result follows by some trivial manipulation of cyclic contexts. The third part is immediate and the finiteness of the set of equivalence classes follows from Lemma 3.1. Firstly, the number of non-pure derived elementary cyclic set is finite. Secondly, for pure elementary cyclic set, the number of equations is uniformly bounded by $\sum_{\phi \in \mathcal{N}(S)} \pi(\phi)$ and the length of the left-hand sides is also bounded by $|c|$, the length of the cycle. □

We are now in position to define the finite automata of an elementary cyclic set S. This automata has as input a sequence σ of developments, member of $\mathcal{A}^\star$. It accepts σ iff it is the sequence of some derived set of S. If S is some elementary cyclic set, its equivalence class is noted $[S]$.

Definition 3.2 *The finite automata $\mathcal{A}(S)$ of an elementary cyclic set S is defined by:*

- *Alphabet: the alphabet $\mathcal{A}$ of the elementary cyclic set S.*
- *States: the equivalence classes of derived sets.*
- *Transitions: there exists a (single) transition from state $[S_1]$ to state $[S_2]$ iff there exists a development from S_1 to S_2.*
- *All states are final, the initial state is $[S]$.*

The definition is sound by Proposition 3.2. Note that these automata are deterministic. The cycle-free part of the automaton, i.e. the set of states s such that there is no non-void transition from s to s, is the set of equivalence classes of non-pure elementary cyclic set. Viewed as a graph, all cycles of the automaton have the same length, equal to $n \times l$. This follows from the definition of the equivalence of two elementary cyclic sets and Lemma 3.1. More precisely, a cycle of the automaton corresponds to an "unfolding" of the cycle from $\mathcal{G}_S$.

These automaton share another interesting property, more conveniently expressed in terms of the set of derived elementary cyclic sets: this set, ordered by the relation $\vdash$ is a lattice.

Lemma 3.3 *The set $\mathcal{L}(S)$ of elementary cyclic sets derived from the elementary cyclic set S is a lattice, under the ordering induced by the relation $\vdash$.*

Proof. Remind that two sequences of developments giving equal elementary cyclic sets (equality defined by equality of the sets of labels) have the same length. This length is the *level* of the derived set S, noted $l(S)$. For σ a substitution that can be decomposed into a sequence of developments, the elementary cyclic set defined from S and σ is noted $s(\sigma)$. We need the following characterization: let σ and σ' be two sequences of developments, then $s(\sigma) = s(\sigma')$ iff $\sigma =_c \sigma'$, where the last equality denotes equality of developments up to permutation. This is easily proved by induction on the (common) length of the developments. We then prove by induction on the length l of the sequences that two operations sup and inf can be defined. Remind that the length of the sequence characterizes the level of the derived set.

This is immediate for $l = 0, 1$ by a diamond chasing. Assume it true for derived sets with level less than l, and let S_1, S_2 be such that $\max(l(S_1), l(S_2)) = l$. Let σ_i, $s(\sigma_i) = S_i$, $i = 1, 2$, be such that the two sequences σ_i share a common prefix σ of maximal length among couples of such sequences of developments. We put

$$S_1 \wedge S_2 =_{def} s(\sigma).$$

We have to establish the independence of this definition with respect to a particular choice of the sequences of developments. Also assume that we have two couples with maximal prefix σ_1, σ_2, σ, and σ_1', σ_2', σ' such that $s(\sigma) \neq s(\sigma')$. We have $|\sigma| = |\sigma'|$. Let τ be the maximal common prefix of σ and σ'. By induction hypothesis, $s'' = s(\tau) = s(\sigma) \wedge s(\sigma')$ is well-defined. From the inequality $s(\sigma) \neq s(\sigma')$, we have, for some development $a \in \mathcal{A}$, $\sigma = \tau a \rho$ and $\sigma' = \tau\lambda$ with $a \notin \lambda$. Let τ_1 and τ_2 be defined by $\sigma_1' = \sigma'\tau_1$ and $\sigma_2' = \sigma'\tau_2$, then a appears in both τ_1 and τ_2 by the fact that a does not occur in λ and the implication $s(\alpha) = s(\beta) \Rightarrow \alpha =_c \beta$. Let $\tau_i = \alpha_i a \beta_i$, α_i a-free, $i = 1, 2$. Then the two sequences $\tau a \lambda \alpha_i \beta_i$, defining S_i, $i = 1, 2$, sharing a prefix of length $|w| + 1$, gives a contradiction.

We construct now the sup of the sets S_1 and S_2. Let $\sigma_i = \sigma\tau_i$, $i = 1, 2$, with $s(\sigma) = S_1 \wedge S_2$. Then we put

$$S_1 \vee S_2 =_{def} \sigma.merge\tau_1, \tau_2)$$

where $merge(\tau_1, \tau_2)$ is defined by:

- $merge(\alpha, \beta) = \alpha\beta$ if α and β do not share any development.
- $merge(a_1 \cdots a_n a \alpha, b_1 \cdots b_m a \beta) = a_1 \cdots a_n a.merge(\alpha, b_1 \cdots b_m \beta)$ if $a_i \notin \beta$ and $b_j \notin \alpha$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

We have to establish the independence of this definition with respect to the representants σ_i. Let $\sigma_i = \sigma\tau_i$ and $\sigma_i' = \sigma\tau_i'$, $i = 1, 2$. We have $\tau_i =_c \tau_i'$, $i = 1, 2$, which implies $merge(\tau_1, \tau_2) =_c merge(\tau_1', \tau_2')$, by an easy induction. Finally, this equation implies $S(\sigma.merge\tau_1, \tau_2)) = S(\sigma.merge\tau_1', \tau_2'))$. To conclude the proof, it is easy to check that these definitions satisfy the usual laws defining a lattice. □

We conclude with two exemples. Needed variables are denoted by greek letters.

$$S \quad \left\{ \begin{array}{lcl} \phi & = & \alpha \to a \\ \psi & = & \beta \to \delta \\ \psi & = & b \to \epsilon \end{array} \right. \qquad \begin{array}{lcl} \phi & = & \psi \\ \alpha & = & \epsilon \to c \\ \delta & = & \beta \to d \end{array}$$

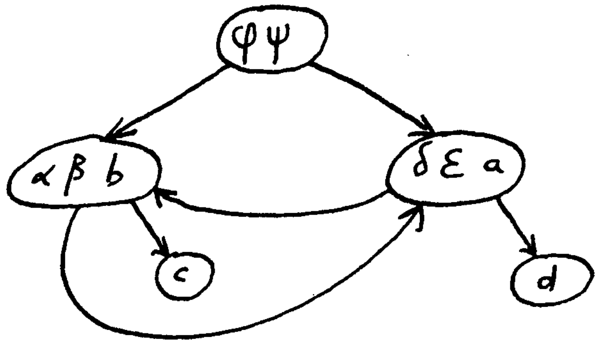

Figure 2: The graph $\mathcal{G}_S$.

We develop once ϕ and ψ into $\phi_1 \to \phi_2$ and $\psi_1 \to \psi_2$ respectively. This gives the elementary cyclic set:

$$S' \begin{cases} \phi_1 &= \alpha \qquad \phi_1 = \psi_1 \\ \psi_1 &= \beta \qquad \alpha = \epsilon \to c \\ \psi_2 &= \delta \qquad \delta = \beta \to d \\ \psi_2 &= \epsilon \end{cases}$$

Note that the variable ϕ_2 does not appear in S', which is coherent with the definition of $\mathcal{A}$.

The last exemple is the following elementary cyclic set S'':

$$\alpha = (\beta \to a) \to b \qquad \beta = (\alpha \to c) \to d$$

Its automata is represented in Figure 3. Notice the existence of sequences of developments, of length $(n-1) \times l + (l-1) \times n$, which does not fully develop neither α nor β along the cycle.

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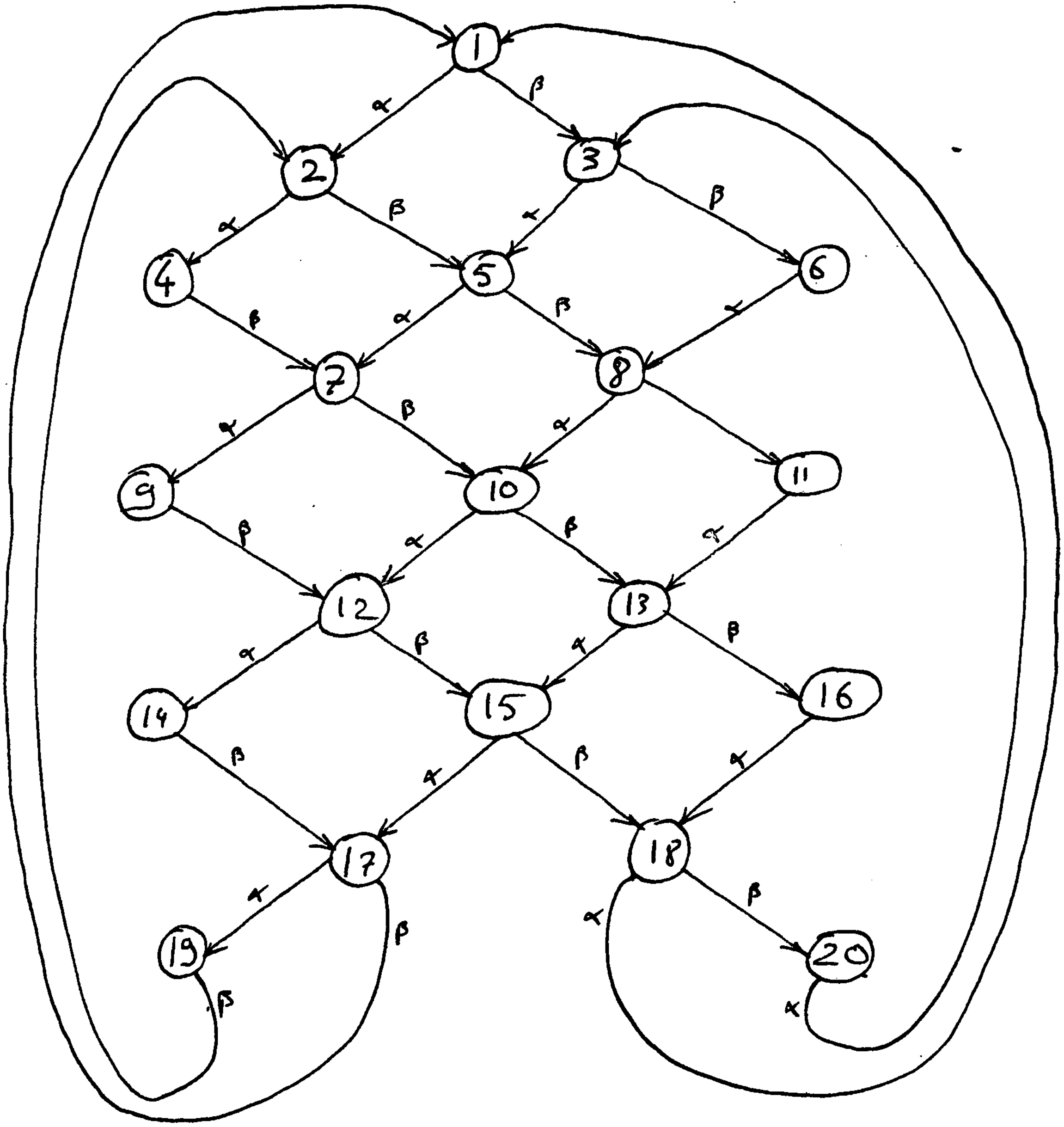

Figure 3: The Automaton $\mathcal{A}(S'')$.

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4 Appendix 1: Reductions for minimum deductions

Case of (t)-rule. Let $\mathcal{D}_0$ be the deduction

$$(t)\,\dfrac{\stackrel{\mathcal{D}_1}{\psi = \phi} \quad \phi = C_0[\theta] \to \tau_1}{\psi = C_0[\theta] \to \tau_1}$$

By the normal form of deductions, this rule is either left premiss of a (d)-rule, left or right premiss of a (su)-rule, or conclusion of the deduction. This gives four subcases.

$$(d)\,\dfrac{\stackrel{\mathcal{D}_0}{\psi = C_0[\theta] \to \tau_1} \quad \stackrel{\mathcal{D}_2}{\psi = C_1[\tau_2] \to \tau_3}}{\theta = \tau_2} \quad \Rightarrow \quad (d)\,\dfrac{\phi_1 = C_0[\theta] \quad (d)\,\dfrac{\stackrel{\overline{\mathcal{D}_1}}{\psi = \phi_1 \to \phi_2} \quad \stackrel{\mathcal{D}_2}{\psi = C_1[\tau_2] \to \tau_3}}{\phi_1 = C_1[\tau_2]}}{\theta = \tau_2}$$

$$(su)\,\dfrac{\stackrel{\mathcal{D}_0}{\psi = C_0[\theta] \to \tau_1} \quad \stackrel{\mathcal{D}_2}{\omega = C_1[\psi]}}{\omega = C_1[C_0[\theta] \to \tau_1]} \quad \Rightarrow \quad (su)\,\dfrac{\phi_1 = C_0[\theta] \quad (su)\,\dfrac{\stackrel{\overline{\mathcal{D}_1}}{\psi = \phi_1 \to \phi_2} \quad \stackrel{\mathcal{D}_2}{\omega = C_1[\psi]}}{\omega = C_1[\phi_1 \to \phi_2]}}{\omega = C_1[C_0[\theta] \to \phi_2]}$$

$$(su)\,\dfrac{\stackrel{\mathcal{D}_2}{\theta = C_1[\omega]} \quad \stackrel{\mathcal{D}_0}{\psi = C_0[\theta] \to \tau_1}}{\psi = C_0[C_1[\omega]] \to \tau_1} \quad \Rightarrow \quad (su)\,\dfrac{\stackrel{\mathcal{D}_2}{\theta = C_1[\omega]} \quad (su)\,\dfrac{\phi_1 = C_0[\theta] \quad \stackrel{\overline{\mathcal{D}_1}}{\psi = \phi_1 \to \phi_2}}{\psi = C_0[\theta] \to \phi_2}}{\psi = C_0[C_1[\omega]] \to \phi_2}$$

$$\stackrel{\mathcal{D}_0}{\psi = C_0[\psi] \to \tau_1} \quad \Rightarrow \quad (su)\,\dfrac{\phi_1 = C_0[\psi] \quad \stackrel{\overline{\mathcal{D}_1}}{\psi = \phi_1 \to \phi_2}}{\psi = C_0[\psi] \to \phi_2}$$

Case of (d)-rule. If the axiom is the left premiss, we have two cases: the other premiss is either an axiom or the conclusion of a (d)-rule.

$$(d)\frac{\phi = C_0[\psi] \to \tau_1 \quad \phi = C_1[\tau_2] \to \tau_3}{\psi = \tau_2} \quad \Rightarrow \quad (d)\frac{\phi_1 = C_0[\psi] \quad \phi_1 = C_1[\tau_2]}{\psi = \tau_2}$$

$$(d)\frac{\phi = C_0[\psi] \to \tau_1 \quad (d)\dfrac{\begin{matrix}\mathcal{D}_2 \\ \omega = C_1[\phi]\end{matrix} \quad \begin{matrix}\mathcal{D}_3 \\ \omega = C_2[C_3[\tau_2] \to \tau_3]\end{matrix}}{\phi = C_3[\tau_2] \to \tau_3}}{\psi = \tau_2} \Rightarrow$$

$$(d)\frac{\phi_1 = C_0[\psi] \quad (d)\dfrac{\begin{matrix}\overline{\mathcal{D}_2} \\ \omega = C_1[\phi_1 \to \phi_2]\end{matrix} \quad \begin{matrix}\mathcal{D}_3 \\ \omega = C_2[C_3[\tau_2] \to \tau_3]\end{matrix}}{\phi_1 = C_3[\tau_2]}}{\psi = \tau_2}$$

If the axiom is the right premiss, the other premiss is either an axiom (subcase considered above) or the conclusion of a (t)- or (d)-rule. In the second case, we need the following

Lemma 4.1 *If $\mathcal{E} \vdash \phi = \psi$ and ϕ is imitated, there exists a deduction $\underline{\mathcal{E}} \vdash \psi = \phi_1 \to \phi_2$.*

Proof. Consider the maximal right branch exclusively composed of (t)-rules above the conclusion $\phi = \psi$ in $\mathcal{E}$: we have

$$(t)\frac{\begin{matrix}\mathcal{D}_0 \\ \phi = \omega_0\end{matrix} \quad \begin{matrix}\mathcal{D}_1 \\ \omega_0 = \omega_1\end{matrix}}{\psi = \omega_1}$$

$$\vdots$$

$$(t)\frac{\psi = \omega_{n-1} \qquad \begin{matrix}\mathcal{D}_n \\ \omega_{n-1} = \psi\end{matrix}}{\phi = \psi}$$

The proof $\mathcal{D}_0$ either reduces to an axiom (with possibly an instance of the symmetry rule) or ends with a (d)-rule. In both cases, we build and reduce the deduction:

$$(t)\frac{(s)\dfrac{\begin{matrix}\mathcal{D}_n \\ \omega_{n-1} = \psi\end{matrix}}{\psi = \omega_{n-1}} \quad (s)\dfrac{\begin{matrix}\mathcal{D}_{n-1} \\ \omega_{n-2} = \omega_{n-1}\end{matrix}}{\omega_{n-1} = \omega_{n-2}}}{\psi = \omega_{n-2}}$$

$$\vdots$$

$$(t)\frac{\psi = \omega_0 \qquad \begin{matrix}\mathcal{D}_0^1 \\ \omega_0 = \phi_1 \to \phi_2\end{matrix}}{\psi = \phi_1 \to \phi_2}$$

where $\mathcal{D}_0^1 \vdash \omega_0 = \phi_1 \to \phi_2$ is the new axiom in the former case or, in the latter, the contractum of the reduction

$$(d)\frac{\begin{matrix}\mathcal{D}_0^2 \\ \omega = C_0[\phi]\end{matrix} \quad \begin{matrix}\mathcal{D}_0^3 \\ \omega = C_1[\omega_0]\end{matrix}}{\phi = \omega_0} \quad \Rightarrow \quad (d)\frac{\begin{matrix}\mathcal{D}_0^3 \\ \omega = C_1[\omega_0]\end{matrix} \quad \begin{matrix}\overline{\mathcal{D}_0^2} \\ \omega = C_0[\phi_1 \to \phi_2]\end{matrix}}{\omega_0 = \phi_1 \to \phi_2} \quad \square$$

Then we have the redexes:

$$(d)\;\dfrac{(t)\;\dfrac{\begin{array}{cc}\mathcal{D}_2 & \mathcal{D}_3\\ \phi=\omega & \omega=C_1[\psi]\to\tau_1\end{array}}{\phi=C_1[\psi]\to\tau_1}\qquad \phi=C_0[\tau_2]\to\tau_3}{\psi=\tau_2}\quad\Rightarrow$$

$$(d)\;\dfrac{(d)\;\dfrac{\begin{array}{cc}\underline{\mathcal{D}_2} & \mathcal{D}_3\\ \omega=\phi_1\to\phi_2 & \omega=C_1[\psi]\to\tau\end{array}}{\phi_1=C_1[\psi]}\qquad \phi_1=C_0[\tau_2]}{\psi=\tau_2}$$

$$(d)\;\dfrac{(d)\;\dfrac{\begin{array}{cc}\mathcal{D}_2 & \mathcal{D}_3\\ \omega=C_1[\phi] & \omega=C_2[C_3[\psi]\to\tau_2]\end{array}}{\phi=C_3[\psi]\to\tau_2}\qquad \phi=C_0[\tau_2]\to\tau_3}{\psi=\tau_2}\quad\Rightarrow$$

$$(d)\;\dfrac{(d)\;\dfrac{\begin{array}{cc}\overline{\mathcal{D}_2} & \mathcal{D}_3\\ \omega=C_1[\phi_1\to\phi_2] & \omega=C_2[C_3[\psi]\to\tau]\end{array}}{\phi_1=C_3[\psi]}\qquad \phi_1=C_0[\tau_2]}{\psi=\tau_2}$$

Case of (su)-rule. When the axiom is the left premiss, we apply the reduction:

$$(su)\;\dfrac{\begin{array}{cc} & \mathcal{D}_1\\ \phi=C_0[\theta]\to\tau_1 & \psi=C_1[\phi]\end{array}}{\psi=C_1[C_0[\theta]\to\tau_1]}\quad\Rightarrow\quad(su)\;\dfrac{\begin{array}{cc} & \overline{\mathcal{D}_1}\\ \phi_1=C_0[\theta] & \psi=C_1[\phi_1\to\phi_2]\end{array}}{\psi=C_1[C_0[\theta]\to\phi_2]}$$

If the axiom is the right premiss:

$$(su)\;\dfrac{\begin{array}{cc}\mathcal{D}_1 & \\ \psi=\tau_1 & \phi=C_0[\psi]\to\tau_2\end{array}}{\phi=C_0[\tau_1]\to\tau_2}\quad\Rightarrow\quad(su)\;\dfrac{\begin{array}{cc}\mathcal{D}_1 & \\ \psi=\tau_1 & \phi_1=C_0[\psi]\end{array}}{\phi_1=C_0[\tau_1]}$$

Notice that in the last redex, the last inference of the deduction $\mathcal{D}$ is a (su)-rule with left premiss of the form $\theta=C[\phi]$, which becomes $\theta=C[\phi_1\to\phi_2]$, and we still have a cyclic deduction.

We now assume that the variable ϕ possesses two right-hand side occurrences eliminated by the same rule. By the results in [10] we know that these occurrences are distinct. Further, we know that ϕ is imitated by a development. Consequently, ϕ possesses at least three distinct occurrences in the axioms. Hence this variable cannot be cyclic. This gives us the following cases.
A (d)-rule eliminates ϕ.

$$(d)\;\dfrac{(t)\;\dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2\\ \phi=\omega & \omega=C_1[\psi]\to\tau_1\end{array}}{\phi=C_1[\psi]\to\tau_1}\qquad (d)\;\dfrac{\begin{array}{cc}\mathcal{D}_3 & \mathcal{D}_4\\ \theta=C_2[\phi] & \theta=C_3[C_4[\tau_2]\to\tau_3]\end{array}}{\phi=C_4[\tau_2]\to\tau_3}}{\psi=\tau_2}\quad\Rightarrow$$

$$(d)\,\dfrac{(d)\,\dfrac{\begin{array}{c}\underline{\mathcal{D}_1}\\ \omega=\phi_1\to\phi_2\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \omega=C_1[\psi]\to\tau_1\end{array}}{\phi_1=C_1[\psi]}\qquad(d)\,\dfrac{\begin{array}{c}\overline{\mathcal{D}_3}\\ \theta=C_2[\phi_1\to\phi_2]\end{array}\quad\begin{array}{c}\mathcal{D}_4\\ \theta=C_3[C_4[\tau_2]\to\tau_3]\end{array}}{\phi_1=C_4[\tau_2]}}{\psi=\tau_2}$$

$$(d)\,\dfrac{(d)\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \omega=C_1[\phi]\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \omega=C_2[C_3[\psi]\to\tau_1]\end{array}}{\phi=C_3[\psi]\to\tau_1}\qquad(d)\,\dfrac{\begin{array}{c}\mathcal{D}_3\\ \theta=C_4[\phi]\end{array}\quad\begin{array}{c}\mathcal{D}_4\\ \theta=C_5[C_6[\tau_2]\to\tau_3]\end{array}}{\phi=C_6[\tau_2]\to\tau_3}}{\psi=\tau_2}\quad\Rightarrow$$

$$(d)\,\dfrac{(d)\,\dfrac{\begin{array}{c}\overline{\mathcal{D}_1}\\ \omega=C_1[\phi_1\to\phi_2]\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \omega=C_2[C_3[\psi]\to\tau_1]\end{array}}{\phi_1=C_3[\psi]}\qquad(d)\,\dfrac{\begin{array}{c}\overline{\mathcal{D}_3}\\ \theta=C_4[\phi_1\to\phi_2]\end{array}\quad\begin{array}{c}\mathcal{D}_4\\ \theta=C_5[C_6[\tau_2]\to\tau_3]\end{array}}{\phi_1=C_6[\tau_2]}}{\psi=\tau_2}$$

A (t)-rule eliminates ϕ.
Firstly, the left premiss is the conclusion of a (t)-rule, and whose right premiss concludes a (d)-rule. In turn, this subdeduction can be left premiss of a (d)-rule, but cannot be left or right premiss of a (su)-rule, nor conclude the deduction $\mathcal{D}$, as ϕ is non-cyclic.

$$(d)\,\dfrac{(t)\,\dfrac{(t)\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi=\omega\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \omega=\phi\end{array}}{\psi=\phi}\qquad(d)\,\dfrac{\begin{array}{c}\mathcal{D}_3\\ \xi=C_1[\phi]\end{array}\quad\begin{array}{c}\mathcal{D}_4\\ \xi=C_2[C_3[\theta]\to\tau_1]\end{array}}{\phi=C_3[\theta]\to\tau_1}}{\psi=C_3[\theta]\to\tau_1}\qquad\begin{array}{c}\mathcal{D}_5\\ \psi=C_4[\tau_2]\to\tau_3\end{array}}{\theta=\tau_2}\quad\Rightarrow$$

$$(d)\,\dfrac{(d)\,\dfrac{\begin{array}{c}\overline{\mathcal{D}_3}\\ \xi=C_1[\phi_1\to\phi_2]\end{array}\quad\begin{array}{c}\mathcal{D}_4\\ \xi=C_2[C_3[\theta]\to\tau_1]\end{array}}{\phi_1=C_3[\theta]}\qquad(d)\,\dfrac{(t)\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \psi=\omega\end{array}\quad\begin{array}{c}\overline{\mathcal{D}_2}\\ \omega=\phi_1\to\phi_2\end{array}}{\psi=\phi_1\to\phi_2}\qquad\begin{array}{c}\mathcal{D}_5\\ \psi=C_4[\tau_2]\to\tau_3\end{array}}{\phi_1=C_4[\tau_2]}}{\theta=\tau_2}$$

If $C_3[\theta]=\theta$ then the last (d)-inference of the contractum is replaced by a (t)-rule and the premisses of the leftmost (d)-rule are permuted.

Secondly, we have the case where the left premiss of the (t)-rule is the conclusion of a (d)-rule.

$$(d)\,\dfrac{(t)\,\dfrac{(d)\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \omega=C_1[\psi]\end{array}\quad\begin{array}{c}\mathcal{D}_2\\ \omega=C_2[\phi]\end{array}}{\psi=\phi}\qquad(d)\,\dfrac{\begin{array}{c}\mathcal{D}_3\\ \xi=C_3[\phi]\end{array}\quad\begin{array}{c}\mathcal{D}_4\\ \xi=C_4[C_5[\theta]\to\tau_1]\end{array}}{\phi=C_5[\theta]\to\tau_1}}{\psi=C_5[\theta]\to\tau_1}\qquad\begin{array}{c}\mathcal{D}_5\\ \psi=C_6[\tau_2]\to\tau_3\end{array}}{\theta=\tau_2}\quad\Rightarrow$$

$$(d)\,\dfrac{(d)\,\dfrac{\begin{array}{c}\overline{\mathcal{D}_3}\\ \xi=C_3[\phi_1\to\phi_2]\end{array}\quad\begin{array}{c}\mathcal{D}_4\\ \xi=C_4[C_5[\theta]\to\tau_1]\end{array}}{\phi_1=C_5[\theta]}\qquad(d)\,\dfrac{(d)\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \omega=C_1[\psi]\end{array}\quad\begin{array}{c}\overline{\mathcal{D}_2}\\ \omega=C_2[\phi_1\to\phi_2]\end{array}}{\psi=\phi_1\to\phi_2}\qquad\begin{array}{c}\mathcal{D}_5\\ \psi=C_6[\tau_2]\to\tau_3\end{array}}{\phi_1=C_6[\tau_2]}}{\theta=\tau_2}$$

The other cases all imply ϕ cyclic, which is impossible.

5 Appendix 2: Reductions for derived deductions lifting

As in the previous Appendix, we implicitely rely on the normal form of deductions for the enumeration of redexes.

Case of 2 axioms in the premisses.

$$(t)\,\frac{\psi = \phi_1 \quad \phi_1 = C[\theta]}{\psi = C[\theta]} \quad \Rightarrow \quad (d)\,\frac{\phi = \psi \to \tau_1 \quad \phi = C[\theta] \to \tau_2}{\psi = C[\theta]}$$

$$(d)\,\frac{\phi_1 = C_0[\theta] \quad \phi_1 = C_1[\tau]}{\theta = \tau} \quad \Rightarrow \quad (d)\,\frac{\phi = C_0[\theta] \to \tau_1 \quad \phi = C_1[\tau] \to \tau_2}{\theta = \tau}$$

$$(su)\,\frac{\phi_1 = C_0[\psi] \quad \theta = C_1[\phi_1 \to \phi_2]}{\theta = C_1[C_0[\psi] \to \phi_1]} \quad \Rightarrow \quad (su)\,\frac{\phi = C_0[\psi] \to \tau \quad \theta = C_1[\phi]}{\theta = C_1[C_0[\psi] \to \tau]}$$

Case of 1 axiom in the premisses.
Rule (t).

$$(t)\,\dfrac{\psi = \phi_1 \qquad (d)\,\dfrac{\begin{matrix}\mathcal{D}_1 \\ \omega = C_0[\phi_1 \to \phi_2]\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \omega = C_1[C_2[\theta] \to \tau]\end{matrix}}{\phi_1 = C_2[\theta]}}{\psi = C_2[\theta]} \quad \Rightarrow$$

$$(d)\,\dfrac{\phi = \psi \to \tau' \qquad (d)\,\dfrac{\begin{matrix}\overline{\mathcal{D}_1} \\ \omega = C_0[\phi]\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \omega = C_1[C_2[\theta] \to \tau]\end{matrix}}{\phi = C_2[\theta] \to \tau}}{\psi = C_2[\theta]}$$

$$(t)\,\dfrac{(t)\,\dfrac{\begin{matrix}\mathcal{D}_1 \\ \psi = \omega\end{matrix} \quad \omega = \phi_1}{\psi = \phi_1} \qquad \phi_1 = C[\theta]}{\psi = C[\theta]} \quad \Rightarrow \quad (t)\,\dfrac{\begin{matrix}\mathcal{D}_1 \\ \psi = \omega\end{matrix} \qquad (d)\,\dfrac{\phi = \omega \to \tau \quad \phi = C[\theta] \to \tau'}{\omega = C[\theta]}}{\psi = C[\theta]}$$

$$(t)\,\dfrac{(t)\,\dfrac{\begin{matrix}\mathcal{D}_1 \\ \psi = \omega\end{matrix} \qquad (d)\,\dfrac{\begin{matrix}\mathcal{D}_2 \\ \xi = C_0[\omega \to \tau]\end{matrix} \quad \begin{matrix}\mathcal{D}_3 \\ \xi = C_1[\phi_1 \to \phi_2]\end{matrix}}{\omega = \phi_1}}{\psi = \phi_1} \qquad \phi_1 = C_2[\theta]}{\psi = C_2[\theta]} \quad \Rightarrow$$

$$(t)\,\dfrac{\begin{matrix}\mathcal{D}_1 \\ \psi = \omega\end{matrix} \qquad (d)\,\dfrac{(d)\,\dfrac{\begin{matrix}\overline{\mathcal{D}_3} \\ \xi = C_1[\phi]\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \xi = C_0[\omega \to \tau]\end{matrix}}{\phi = \omega \to \tau} \qquad \phi = C_2[\theta] \to \tau'}{\omega = C_2[\theta]}}{\psi = C_2[\theta]}$$

$$(t)\,\dfrac{(d)\,\dfrac{\overset{\mathcal{D}_1}{\omega = C_0[\psi \to \tau]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\phi_1 \to \phi_2]}}{\psi = \phi_1} \qquad \phi_1 = C_2[\theta]}{\psi = C_2[\theta]} \quad \Rightarrow$$

$$(d)\,\dfrac{(d)\,\dfrac{\overset{\overline{\mathcal{D}_2}}{\omega = C_1[\phi]} \quad \overset{\mathcal{D}_1}{\omega = C_0[\psi \to \tau]}}{\phi = \psi \to \tau} \qquad \phi = C_2[\theta] \to \tau'}{\psi = C_2[\theta]}$$

Rule (d).

$$(d)\,\dfrac{\phi_1 = C_0[\theta] \qquad (d)\,\dfrac{\overset{\mathcal{D}_1}{\omega = C_1[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_2}{\omega = C_2[C_3[\tau] \to \tau']}}{\phi_1 = C_3[\tau]}}{\theta = \tau} \quad \Rightarrow$$

$$(d)\,\dfrac{\phi = C_0[\theta] \to \tau'' \qquad (d)\,\dfrac{\overset{\overline{\mathcal{D}_1}}{\omega = C_1[\phi]} \quad \overset{\mathcal{D}_2}{\omega = C_2[C_3[\tau] \to \tau']}}{\phi = C_3[\tau] \to \tau'}}{\theta = \tau}$$

The next reductions uses the following

Lemma 5.1 *Let S be some elementary cyclic set and σ a development of σ. Assume that $\mathcal{D} \vdash \phi_1 = \omega$ or $\mathcal{D} \vdash \omega = \phi_1$ is a deduction whose axioms belong to* Simpl$(\sigma(S))$. *Then there exists two deductions $\mathcal{D}^1 \vdash \phi = \omega_0 \to \tau$ and $\mathcal{D}^2 \vdash \omega = \omega_0$ or $\mathcal{D}^2 \vdash \omega_0 = \omega$, whose axioms belong to S.*

Proof. The deduction $\mathcal{D}_1$ is of the form

$$\begin{array}{c}
(t)\,\dfrac{\overset{\mathcal{D}^0}{\phi_1 = \omega_0} \quad \overset{\mathcal{D}^1}{\omega_0 = \omega_1}}{\phi_1 = \omega_1} \\
\vdots \\
(t)\,\dfrac{\phi_1 = \omega_{n-1} \qquad \overset{\mathcal{D}^n}{\omega_{n-1} = \omega}}{\phi_1 = \omega}
\end{array}$$

where $\mathcal{D}^0$ is either an axiom, in which case $\phi = \omega_0 \to \tau$ belongs to S for some term τ, or ends with a (d)-rule, in which case we introduce

$$(d)\,\dfrac{\overset{\mathcal{E}_0}{\psi = C_0[\phi_1 \to \phi_2]} \quad \overset{\mathcal{E}_1}{\psi = C_1[\omega_0 \to \tau]}}{\phi_1 = \omega_0} \quad \Rightarrow \quad (d)\,\dfrac{\overset{\overline{\mathcal{E}_0}}{\psi = C_0[\phi]} \quad \overset{\mathcal{E}_1}{\psi = C_1[\omega_0 \to \tau]}}{\phi = \omega_0 \to \tau}$$

Finally, the inferences $\mathcal{D}^i$, $i = 0, \ldots, n$, prove $\omega = \omega_0$ and $\omega_0 = \omega$. The case $\mathcal{D} \vdash \omega = \phi_1$ is similar.
□

In some of the following reductions, we apply the lemma with the same notational conventions as in its statement.

$$(d)\,\frac{\phi_1 = C_0[\theta] \qquad (t)\,\dfrac{\stackrel{\mathcal{D}_1}{\phi_1 = \omega} \quad \stackrel{\mathcal{D}_2}{\omega = C_1[\tau]}}{\phi_1 = C_1[\tau]}}{\theta = \tau} \quad \Rightarrow$$

$$(d)\,\frac{(t)\,\dfrac{\stackrel{\mathcal{D}_1^2}{\omega = \omega_0} \qquad (d)\,\dfrac{\stackrel{\mathcal{D}_1^1}{\phi = \omega_0 \to \tau' \quad \phi = C_0[\theta] \to \tau''}}{\omega_0 = C_0[\theta]}}{\omega = C_0[\theta]} \qquad \stackrel{\mathcal{D}_2}{\omega = C_1[\tau]}}{\theta = \tau}$$

$$(d)\,\frac{(t)\,\dfrac{\stackrel{\mathcal{D}_1}{\phi_1 = \omega} \quad \stackrel{\mathcal{D}_2}{\omega = C_0[\theta]}}{\phi_1 = C_0[\theta]} \qquad \phi_1 = C_1[\tau]}{\theta = \tau} \quad \Rightarrow$$

$$(d)\,\frac{\stackrel{\mathcal{D}_2}{\omega = C_0[\theta]} \qquad (t)\,\dfrac{\stackrel{\mathcal{D}_1^2}{\omega = \omega_0} \qquad (d)\,\dfrac{\stackrel{\mathcal{D}_1^1}{\phi = \omega_0 \to \tau \quad \phi = C_1[\tau] \to \tau'}}{\omega_0 = C_1[\tau]}}{\omega = C_1[\tau]}}{\theta = \tau}$$

$$(d)\,\frac{(d)\,\dfrac{\stackrel{\mathcal{D}_1}{\psi = C_0[\phi_1 \to \phi_2]} \quad \stackrel{\mathcal{D}_2}{\psi = C_1[C_2[\theta] \to \tau']}}{\phi_1 = C_2[\theta]} \qquad \phi_1 = C_3[\tau]}{\theta = \tau} \quad \Rightarrow$$

$$(d)\,\frac{(d)\,\dfrac{\stackrel{\overline{\mathcal{D}_1}}{\psi = C_0[\phi]} \quad \stackrel{\mathcal{D}_2}{\psi = C_1[C_2[\theta] \to \tau']}}{\phi = C_2[\theta] \to \tau'} \qquad \phi = C_3[\tau] \to \tau''}{\theta = \tau}$$

Rule (su).

$$(su)\,\frac{\phi_1 = \tau \qquad (su)\,\dfrac{\stackrel{\mathcal{D}_1}{\omega = C_1[\phi_1 \to \phi_2]} \quad \stackrel{\mathcal{D}_2}{\theta = C_2[\omega]}}{\theta = C_2[C_1[\phi_1 \to \phi_2]]}}{\theta = C_2[C_1[\tau \to \phi_2]]} \quad \Rightarrow$$

$$(su)\,\frac{\phi = \tau \to \tau' \qquad (su)\,\dfrac{\stackrel{\overline{\mathcal{D}_1}}{\omega = C_1[\phi]} \quad \stackrel{\mathcal{D}_2}{\theta = C_2[\omega]}}{\theta = C_2[C_1[\phi]]}}{\theta = C_2[C_1[\tau \to \tau']]}$$

$$(su)\,\dfrac{\phi_1=\tau \qquad (d)\,\dfrac{\overset{\mathcal{D}_1}{\omega=C_0[\theta]} \quad \overset{\mathcal{D}_2}{\omega=C_1[C_2[\phi_1\to\phi_2]]}}{\theta=C_2[\phi_1\to\phi_2]}}{\theta=C_2[\tau\to\phi_2]} \quad \Rightarrow$$

$$(su)\,\dfrac{\phi=\tau\to\tau' \qquad (d)\,\dfrac{\overset{\mathcal{D}_1}{\omega=C_0[\theta]} \quad \overset{\overline{\mathcal{D}_2}}{\omega=C_1[C_2[\phi]]}}{\theta=C_2[\phi]}}{\theta=C_2[\tau\to\tau']}$$

$$(su)\,\dfrac{\phi_1=\tau \qquad (t)\,\dfrac{\overset{\mathcal{D}_1}{\theta=\omega} \quad \overset{\mathcal{D}_2}{\omega=C[\phi_1\to\phi_2]}}{\theta=C[\phi_1\to\phi_2]}}{\theta=C[\tau\to\phi_2]} \quad \Rightarrow \quad (su)\,\dfrac{\phi=\tau\to\tau' \qquad (t)\,\dfrac{\overset{\mathcal{D}_1}{\theta=\omega} \quad \overset{\overline{\mathcal{D}_2}}{\omega=C[\phi]}}{\theta=C[\phi]}}{\theta=C[\tau\to\tau']}$$

$$(su)\,\dfrac{(d)\,\dfrac{\overset{\mathcal{D}_1}{\omega=C_0[\phi_1\to\phi_2]} \quad \overset{\mathcal{D}_2}{\omega=C_1[\tau\to\tau']}}{\phi_1=\tau} \qquad \theta=C_2[\phi_1\to\phi_2]}{\theta=C_2[\tau\to\phi_2]} \quad \Rightarrow$$

$$(su)\,\dfrac{(d)\,\dfrac{\overset{\overline{\mathcal{D}_1}}{\omega=C_0[\phi]} \quad \overset{\mathcal{D}_2}{\omega=C_1[\tau\to\tau']}}{\phi=\tau\to\tau'} \qquad \theta=C_2[\phi]}{\theta=C_2[\tau\to\tau']}$$

$$(su)\,\dfrac{(t)\,\dfrac{\overset{\mathcal{D}_1}{\phi_1=\psi} \quad \overset{\mathcal{D}_2}{\psi=\tau}}{\phi_1=\tau} \qquad \theta=C[\phi_1\to\phi_2]}{\theta=C[\tau\to\phi_2]} \quad \Rightarrow$$

$$(su)\,\dfrac{(t)\,\dfrac{\overset{\mathcal{D}_1^2}{\omega_0=\psi} \quad \overset{\mathcal{D}_2}{\psi=\tau}}{\omega_0=\tau} \qquad (su)\,\dfrac{\overset{\mathcal{D}_1^1}{\phi=\omega_0\to\tau'} \quad \theta=C[\phi]}{\theta=C[\omega_0\to\tau]}}{\theta=C[\tau\to\tau']}$$

Case of non-axiom premisses.
Rule (t).

$$(t)\,\dfrac{(t)\,\dfrac{\overset{\mathcal{D}_1}{\psi=\omega} \quad \overset{\mathcal{D}_2}{\omega=\phi_1}}{\psi=\phi_1} \qquad (d)\,\dfrac{\overset{\mathcal{D}_3}{\theta=C_0[\phi_1\to\phi_2]} \quad \overset{\mathcal{D}_4}{\theta=C_1[\tau\to\tau']}}{\phi_1=\tau}}{\psi=\tau} \quad \Rightarrow$$

$$(t)\,\dfrac{(t)\,\dfrac{\overset{\mathcal{D}_1}{\psi=\omega} \quad \overset{\mathcal{D}_2^1}{\omega=\omega_0}}{\psi=\omega_0} \qquad (d)\,\dfrac{\overset{\mathcal{D}_2^2}{\phi=\omega_0\to\tau''} \quad (d)\,\dfrac{\overset{\overline{\mathcal{D}_3}}{\theta=C_0[\phi]} \quad \overset{\mathcal{D}_4}{\theta=C_1[\tau\to\tau']}}{\phi=\tau\to\tau'}}{\omega_0=\tau}}{\psi=\tau}$$

$$
(t)\;\dfrac{(d)\;\dfrac{\overset{\mathcal{D}_1}{\omega = C_0[\psi \to \tau]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\phi_1 \to \phi_2]}}{\psi = \phi_1} \qquad (d)\;\dfrac{\overset{\mathcal{D}_3}{\theta = C_2[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_4}{\theta = C_3[\tau \to \tau']}}{\phi_1 = \tau}}{\psi = \tau} \quad \Rightarrow
$$

$$
(t)\;\dfrac{(d)\;\dfrac{\overset{\overline{\mathcal{D}_2}}{\omega = C_1[\phi]} \quad \overset{\mathcal{D}_1}{\omega = C_0[\psi \to \tau]}}{\phi = \psi \to \tau} \qquad (d)\;\dfrac{\overset{\overline{\mathcal{D}_3}}{\theta = C_2[\phi]} \quad \overset{\mathcal{D}_4}{\theta = C_3[\tau \to \tau']}}{\phi = \tau \to \tau'}}{\psi = \tau}
$$

Rule (d).

$$
(d)\;\dfrac{(d)\;\dfrac{\overset{\mathcal{D}_1}{\omega = C_2[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_2}{\omega = C_3[C_0[\psi] \to \tau']}}{\phi_1 = C_0[\psi]} \qquad (d)\;\dfrac{\overset{\mathcal{D}_3}{\theta = C_4[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_4}{\theta = C_5[C_1[\tau] \to \tau'']}}{\phi_1 = C_1[\tau]}}{\psi = \tau} \quad \Rightarrow
$$

$$
(d)\;\dfrac{(d)\;\dfrac{\overset{\overline{\mathcal{D}_1}}{\omega = C_2[\phi]} \quad \overset{\mathcal{D}_2}{\omega = C_3[C_0[\psi] \to \tau']}}{\phi = C_0[\psi] \to \tau'} \qquad (d)\;\dfrac{\overset{\overline{\mathcal{D}_3}}{\theta = C_4[\phi]} \quad \overset{\mathcal{D}_4}{\theta = C_5[C_1[\tau] \to \tau'']}}{\phi = C_1[\tau] \to \tau''}}{\psi = \tau}
$$

$$
(d)\;\dfrac{(t)\;\dfrac{\overset{\mathcal{D}_1}{\phi_1 = \omega} \quad \overset{\mathcal{D}_2}{\omega = C_0[\psi]}}{\phi_1 = C_0[\psi]} \qquad (d)\;\dfrac{\overset{\mathcal{D}_3}{\theta = C_2[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_4}{\theta = C_3[C_1[\tau] \to \tau']}}{\phi_1 = C_1[\tau]}}{\psi = \tau} \quad \Rightarrow
$$

$$
(d)\;\dfrac{\overset{\mathcal{D}_2}{\omega = C_0[\psi]} \quad (t)\;\dfrac{\overset{\mathcal{D}_1^2}{\omega = \omega_0} \quad (d)\;\dfrac{\overset{\mathcal{D}_1^1}{\phi = \omega_0 \to \tau} \quad (d)\;\dfrac{\overset{\overline{\mathcal{D}_3}}{\theta = C_2[\phi]} \quad \overset{\mathcal{D}_4}{\theta = C_3[C_1[\tau] \to \tau']}}{\phi = C_1[\tau] \to \tau'}}{\omega_0 = C_1[\tau]}}{\omega = C_1[\tau]}}{\psi = \tau}
$$

Rule (su).

$$
(su)\;\dfrac{(t)\;\dfrac{\overset{\mathcal{D}_1}{\phi_1 = \omega} \quad \overset{\mathcal{D}_2}{\omega = \tau}}{\phi_1 = \tau} \qquad (su)\;\dfrac{\overset{\mathcal{D}_3}{\psi = C_0[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_4}{\theta = C_1[\psi]}}{\theta = C_1[C_0[\phi_1 \to \phi_2]]}}{\theta = C_1[C_0[\tau \to \phi_2]]} \quad \Rightarrow
$$

$$
(su)\;\dfrac{(t)\;\dfrac{\overset{\mathcal{D}_1^2}{\omega_0 = \omega} \quad \overset{\mathcal{D}_2}{\omega = \tau}}{\omega_0 = \tau} \qquad (su)\;\dfrac{\overset{\mathcal{D}_1^1}{\phi = \omega_0 \to \tau'} \quad (su)\;\dfrac{\overset{\overline{\mathcal{D}_3}}{\psi = C_0[\phi]} \quad \overset{\mathcal{D}_4}{\theta = C_1[\psi]}}{\theta = C_1[C_0[\phi]]}}{\theta = C_1[C_0[\omega_0 \to \tau']]}}{\theta = C_1[C_0[\tau \to \tau']]}
$$

$$
(su)\;\dfrac{(d)\;\dfrac{\overset{\mathcal{D}_1}{\omega = C_0[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\tau \to \tau']}}{\phi_1 = \tau} \qquad (su)\;\dfrac{\overset{\mathcal{D}_3}{\psi = C_2[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_4}{\theta = C_3[\psi]}}{\theta = C_3[C_2[\phi_1 \to \phi_2]]}}{\theta = C_3[C_2[\tau \to \phi_2]]} \quad \Rightarrow
$$

$$
(su)\;\dfrac{(d)\;\dfrac{\overset{\overline{\mathcal{D}_1}}{\omega = C_0[\phi]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\tau \to \tau']}}{\phi = \tau \to \tau'} \qquad (su)\;\dfrac{\overset{\overline{\mathcal{D}_3}}{\psi = C_2[\phi]} \quad \overset{\mathcal{D}_4}{\theta = C_3[\psi]}}{\theta = C_3[C_2[\phi]]}}{\theta = C_3[C_2[\tau \to \tau']]}
$$

$$
(su)\;\dfrac{(t)\;\dfrac{\overset{\mathcal{D}_1}{\phi_1 = \omega} \quad \overset{\mathcal{D}_2}{\omega = \tau}}{\phi_1 = \tau} \qquad (d)\;\dfrac{\overset{\mathcal{D}_3}{\psi = C_0[\theta]} \quad \overset{\mathcal{D}_4}{\theta = C_1[C_2[\phi_1 \to \phi_2]]}}{\theta = C_2[\phi_1 \to \phi_2]}}{\theta = C_2[\tau \to \phi_2]} \quad \Rightarrow
$$

$$
(su)\;\dfrac{(t)\;\dfrac{\overset{\mathcal{D}_1^2}{\omega_0 = \omega} \quad \overset{\mathcal{D}_2}{\omega = \tau}}{\omega_0 = \tau} \qquad (su)\;\dfrac{\overset{\mathcal{D}_1^1}{\phi = \omega_0 \to \tau'} \quad (d)\;\dfrac{\overset{\mathcal{D}_3}{\psi = C_0[\theta]} \quad \overset{\overline{\mathcal{D}_4}}{\psi = C_1[C_2[\phi]]}}{\theta = C_2[\phi]}}{\theta = C_2[\omega_0 \to \tau']}}{\theta = C_2[\tau \to \tau']}
$$

$$
(su)\;\dfrac{(d)\;\dfrac{\overset{\mathcal{D}_1}{\omega = C_0[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\tau \to \tau']}}{\phi_1 = \tau} \qquad (d)\;\dfrac{\overset{\mathcal{D}_3}{\psi = C_2[\theta]} \quad \overset{\mathcal{D}_4}{\psi = C_3[C_4[\phi_1 \to \phi_2]]}}{\theta = C_4[\phi_1 \to \phi_2]}}{\theta = C_4[\tau \to \phi_2]} \quad \Rightarrow
$$

$$
(su)\;\dfrac{(d)\;\dfrac{\overset{\overline{\mathcal{D}_1}}{\omega = C_0[\phi]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\tau \to \tau']}}{\phi = \tau \to \tau'} \qquad (d)\;\dfrac{\overset{\mathcal{D}_3}{\psi = C_2[\theta]} \quad \overset{\overline{\mathcal{D}_4}}{\theta = C_3[C_4[\phi]]}}{\theta = C_4[\phi]}}{\theta = C_4[\tau \to \tau']}
$$

$$
(su)\;\dfrac{(t)\;\dfrac{\overset{\mathcal{D}_1}{\phi_1 = \omega} \quad \overset{\mathcal{D}_2}{\omega = \tau}}{\phi_1 = \tau} \qquad (t)\;\dfrac{\overset{\mathcal{D}_3}{\theta = \psi} \quad \overset{\mathcal{D}_4}{\psi = C[\phi_1 \to \phi_2]}}{\theta = C[\phi_1 \to \phi_2]}}{\theta = C[\tau \to \phi_2]} \quad \Rightarrow
$$

$$
(su)\;\dfrac{(t)\;\dfrac{\overset{\mathcal{D}_1^2}{\omega_0 = \omega} \quad \overset{\mathcal{D}_2}{\omega = \tau}}{\omega_0 = \tau} \qquad (su)\;\dfrac{\overset{\mathcal{D}_1^1}{\phi = \omega_0 \to \tau'} \quad (t)\;\dfrac{\overset{\mathcal{D}_3}{\theta = \psi} \quad \overset{\overline{\mathcal{D}_4}}{\psi = C[\phi]}}{\theta = C[\phi]}}{\theta = C[\omega_0 \to \tau']}}{\theta = C[\tau \to \tau']}
$$

$$
(su)\;\dfrac{(d)\;\dfrac{\overset{\mathcal{D}_1}{\omega = C_0[\phi_1 \to \phi_2]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\tau \to \tau']}}{\phi_1 = \tau} \qquad (t)\;\dfrac{\overset{\mathcal{D}_3}{\theta = \psi} \quad \overset{\mathcal{D}_4}{\psi = C_2[\phi_1 \to \phi_2]}}{\theta = C_2[\phi_1 \to \phi_2]}}{\theta = C_2[\tau \to \phi_2]} \quad \Rightarrow
$$

$$(su)\;\dfrac{(d)\;\dfrac{\overset{\overline{\mathcal{D}_1}}{\omega = C_0[\phi]} \quad \overset{\mathcal{D}_2}{\omega = C_1[\tau \to \tau']}}{\phi = \tau \to \tau'} \qquad (t)\;\dfrac{\overset{\mathcal{D}_3}{\theta = \psi} \quad \overset{\overline{\mathcal{D}_4}}{\psi = C_2[\phi]}}{\theta = C_2[\phi]}}{\theta = C_2[\tau \to \tau']}$$

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique